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Explicit expressions for the topological defects of spinor Bose–Einstein condensates

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Abstract

In this paper we first derive a general method which enables one to create expressions for vortices and monopoles. By using this method we construct several order-parameters describing the vortices and monopoles of Bose–Einstein condensates with hyperfine spin $F = 1$ and $F = 2$. We concentrate on defects which are topologically stable in the absence of an external magnetic field. In particular we show that in a ferromagnetic condensate there can be a vortex which does produce any superfluid flow. We also point out that the order-parameter space of the cyclic phase of $F = 2$ condensate consists of two disconnected sets. Finally we examine the effect of an external magnetic field on the vortices of a ferromagnetic $F = 1$ condensate and discuss the experimental preparation of a vortex in this system.

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1. Introduction

During the last ten years Bose–Einstein condensates (BECs) of alkali atoms have turned out to be an excellent system to create and observe several interesting phenomena, such as topological defects [1, 2]. The best-known topological defect is a vortex, which in a typical single component BEC appears as a long-lived line-like singularity in the particle density. In a non-rotating trap, a vortex state cannot be energetically the ground state of the system, but its decay is prevented by topological reasons. The continuous deformations of the order-parameter which are needed in order to reach the ground state require more energy than what is available from e.g. thermal excitations. In the presence of dissipation the vortex can move to the boundary of the condensate and vanish, but even then it is stable as long as it stays in the condensate. The creation of spinor BECs has made it possible to have more complicated vortices and other topological structures than what are allowed by a single component condensate [2–4]. By spinor condensates we mean BECs which have all spin components trapped simultaneously and where spin dynamics between different spin components is possible. For these reasons spinor condensates allow for richer topological structures than single component condensates. In experiments spinor condensates are realized by using an optical trap to trap the condensed atoms. If a BEC is in a magnetic trap, only particles which are in a low-field seeking state with respect to the quantization axis determined by the local magnetic field remain trapped. When an optical trap is used there can still be magnetic fields present. They are not needed to trap atoms, but e.g. to diminish the effect of stray magnetic fields [5–8].

The existence of the topological defects is based on the fact that a BEC can be described by an order-parameter ψ. The order-parameter is a map from some region of physical space into the order-parameter space M. By examining the properties of the order-parameter space one can see what kind of, if any, topological defects are possible. This examination can be carried out with the help of the homotopy groups of the order-parameter space.

The use of the theory of homotopy groups to characterize the topological defects of physical systems was first used during the late fifties [9], but became widely known only in the seventies; see e.g. [10]. Since then it has been successfully applied in several fields of physics, such as condensed matter physics, particle physics and cosmology [10–15]. Homotopy groups classify maps which can be continuously deformed into one another. Homotopy is a mathematical notion giving an exact meaning for this kind of deformation. Homotopy groups have turned out to be an effective way of characterizing and classifying topological defects. This classification can be achieved if the order-parameter space M is identified with a quotient space G/H, where G is a group that acts transitively on the order-parameter space and H is a suitably chosen subgroup of G.

If G/H is known, information on the topological defects can be obtained by calculating the homotopy groups $\pi_n(G/H)$, $n = 0, 1, 2, 3$. From these $\pi_0(G/H)$ characterizes domain walls, $\pi_1(G/H)$ vortices and one-dimensional non-singular defects, $\pi_2(G/H)$ monopoles and two-dimensional non-singular defects, and $\pi_3(G/H)$ three-dimensional non-singular defects. Domain walls, vortices and monopoles are defects where the order-parameter has to vanish at some point of the physical space. The non-singular defects are defects where the order-parameter is nonzero everywhere, and the topological stability is imposed by the boundary conditions. The elements of the homotopy group label the order-parameters in such a way that those labelled by the same group element can be continuously converted into one another, whereas if the configurations are labelled by different group elements, this is not possible. If $\pi_n(G/H) = \{e\}$, i.e. the nth homotopy group is a one-element group, no topologically stable defects characterized by $\pi_n(G/H)$ are possible.

This paper is organized as follows. In section 2 we derive a systematic way to find expressions for vortices and monopoles. This method is based on the properties of the relative homotopy groups, and it has not been presented before. In section 3 we review the properties of spinor condensates and their ground states. As a new result we show that the order-parameter space of the cyclic phase of $F = 2$ condensate consists of two sets which are disconnected. This is in contrast to the order-parameter spaces of other ground-state phases, which consist of one connected set. In section 4 the method derived in section 2 is applied in the context of spinor BECs to create order-parameters describing vortices and monopoles. Most of the expressions for defects have not been presented before. These are then used to find the minimum energy states of the defects and to study the superfluid velocity and angular momentum induced by them. In particular we show that in a ferromagnetic condensate the presence of a vortex does not have to lead to superfluid flow. In section 5 we show that the vortex of a ferromagnetic $F = 1$ condensate derived in section 4 can exist also if the conservation of magnetization is taken into attention. Additionally we show that in a ferromagnetic $F = 1$ condensate the conservation of magnetization may lead to stabilization of defects which are not stable if the magnetization is allowed to vary freely. In section 6 we propose a way to create vortices in a ferromagnetic condensate. These vortices are stable in the absence of an external magnetic field. This way is based on the use of a topological vortex creation method used before to create defects which are stable in the presence of an external field. Finally in section 7 we summarize the results of the paper and in appendix A we give the spin and rotation matrices needed in the calculation of defects.

2. Finding expressions for vortices and monopoles

2.1. Introduction

In this section we develop a method for finding explicit expressions for vortices and monopoles. The content of this section is quite mathematical, and those interested only in the applications may jump to the next section. We briefly introduce some of the necessary concepts and fix the notation. To save space we do not define all mathematical concepts needed; an interested reader can find the definitions in any standard book on homotopy theory, such as [16]. We also omit all proofs here; those can be found in [10, 17]. From now on all maps are assumed to be continuous. In the following G is a Lie group and H is a closed subgroup of G. The subgroup H can be written as $H = H_0 \cup H_1 \cup H_2 \cup \cdots$, where H_i's are the path components of H. The path component containing the identity element e is denoted by H_0. It is a normal subgroup of H and thus the quotient space H/H_0 is a group. The path components of H are cosets of H in H_0, i.e. $H_k = h_k H_0$ for some $h_k \in H$. We define the n-dimensional disc by $D^n = \{\mathbf{x} \in \mathbb{R}^n | |\mathbf{x}| \leqslant 1\}$ and the n-sphere by $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} | |\mathbf{x}| = 1\}$. We denote the nth relative homotopy by $\pi_n(G, H, e)$ and an element of this group by $[f]$, where $f : (D^n, S^{n-1}, s_0) \to (G, H, e)$ is a map. Here the notation means that D^n is taken to G, S^{n-1} to H, and $s_0 \in S^{n-1}$ to e by f. By $[f]$ we denote the equivalence class determined by f. It consists of all maps which are homotopic via maps of this type. If $H = e$, we write $\pi_n(G, e) \equiv \pi_n(G, e, e)$. If G is path connected, $\pi_n(G, g_1)$ and $\pi_n(G, g_2)$ are isomorphic for all $g_1, g_2 \in G$, and we use the notation $\pi_n(G)$ to denote any $\pi_n(G, g_1)$. There is an exact sequence of homomorphisms between relative homotopy groups. This sequence reads

$$\begin{array}{ccccccccc} \xrightarrow{\alpha_n} & \pi_n(G, e) & \xrightarrow{\beta_n} & \pi_n(G, H, e) & \xrightarrow{\gamma_n} & \pi_{n-1}(H, e) & \xrightarrow{\alpha_{n-1}} & \pi_{n-1}(G, e) & \xrightarrow{\beta_{n-1}} \cdots . \\ & & & \downarrow p_* & & & & & \\ & & & \pi_n(G/H, H) & & & & & \end{array} \quad (1)$$

We have included in the sequence p_*, which is an isomorphism determined by the map $p : G \to G/H, g \mapsto gH$. If $[f] \in \pi_n(G, H, e)$ then $p_*([f]) = [pf] \in \pi_n(G/H, H)$.

2.2. Physical applications

Next we consider how previous results can be applied in physical systems. Now G is a group that acts transitively on the order-parameter space M. The action of $g \in G$ on $x \in M$ is denoted by $g \cdot x$. We now choose an arbitrary element $x_{\mathrm{ref}} \in M$, called the reference order-parameter, and define $H = \{g \in G | g \cdot x_{\mathrm{ref}} = x_{\mathrm{ref}}\}$. This is the isotropy group and it is a closed subgroup of G. The order-parameter space M can then be identified with G/H. The correspondence between the elements of M and G/H is $x \Leftrightarrow gH, g \cdot x_{\mathrm{ref}} = x$. If A is a topological space and $f : A \to G$ is a map, then the map $pf : A \to G/H$ gives a map from A to the order-parameter space, which is now represented by G/H. If the order-parameter space is represented by M the corresponding map from A to M is given by $c : A \to M$ such that $c(a) = f(a) \cdot x_{\mathrm{ref}}$ for all $a \in A$. When discussing the physical applications we assume that $\pi_2(G, e) = \pi_1(G, e) = \pi_0(G, e) = \{e\}$, when γ_1 and γ_2 become isomorphisms. These conditions hold for $\mathbb{R}$ and $SU(2)$, which are the groups used in this paper.

2.2.1. Monopoles. Because $\pi_1(H, e) = \pi_1(H_0, e)$, we see that $\pi_2(G, H, e)$ and $\pi_1(H_0, e)$ are isomorphic via γ_2. The map $\gamma_2 p_*^{-1}$ gives an isomorphism between $\pi_2(G/H, H)$ and $\pi_1(H_0, e)$. This isomorphism can be used in the calculation of $\pi_2(G/H, H)$, since usually it is quite easy to see what $\pi_1(H_0, e)$ is [3, 10, 11]. Assume that $[f] \in \pi_2(G, H, e)$ and that $f|_{S^1}$ is the restriction of f to S^1, the boundary of D^2. Then $[f|_{S^1}] \in \pi_1(H_0, e)$. The isomorphism γ_2 is given by the map $[f] \mapsto [f|_{S^1}]$. We use (θ, φ) as the coordinates of D^2, that is, θ is the distance from the centre of the disc and φ is the azimuthal angle. We take the radius of D^2 to be π. Let $g : [0, 2\pi] \to H_0$ be a map for which $g(0) = g(2\pi) = e$ and let $\tilde{g} : D^2 \to G$ be such that $\tilde{g}(\theta = \pi, \varphi) = g(\varphi)$. If we also define $s_0 = (\pi, 0)$, then $[\tilde{g}] \in \pi_2(G, H_0, e)$, $[p\tilde{g}] \in \pi_2(G/H, H)$ and $[\tilde{g}|_{S^1}] = [\tilde{g}|_{\theta=\pi}] = [g] \in \pi_1(H_0, e)$. As explained above, $[pg]$ and $[g]$ are mapped into each other by isomorphism $\gamma_2 p_*^{-1}$.

Next we construct a map from physical space into the order-parameter space G/H which describes a monopole with some given winding number. Now we use spherical coordinates (r, θ, φ) as the coordinates of the physical space $\mathbb{R}^3$ and assume that the monopole is located at the origin. The monopole we construct is independent of the r-coordinate. This assumption is not necessary, but to avoid further complication we use it here. Let $A = \mathbb{R}^3 \setminus \{\mathbf{0}\}$. We define $f : A \to G/H$ by $f(r, \theta, \varphi) = \tilde{g}(\theta, \varphi)H$. Then for each fixed $r > 0\, [f] \in \pi_2(G/H, H)$ and $[f]$ is the unique inverse image of $[g] \in \pi_1(H_0, e)$ in the map $\gamma_2 p_*^{-1}$. These elements have the same winding number. Thus if one wants a map $f : A \to G/H$, which describes a monopole with a given winding number, he has to find $\tilde{g} : D^2 \to G$ such that $[\tilde{g}|_{\theta=\pi}]$ is an element of $\pi_1(H_0, e)$ with the wanted winding number. Then f is obtained by defining $f(r, \theta, \varphi) = \tilde{g}(\theta, \varphi)H$. The corresponding order-parameter $x : A \to M$ is defined by $x(r, \theta, \varphi) = \tilde{g}(\theta, \varphi) \cdot x_{\text{ref}}$.

2.2.2. Vortices. The group structure of $\pi_0(H, e)$ has to be defined a little differently than that of other relative homotopy groups [10]. We define $\pi_0(H, e) \equiv H/H_0 = \{H_0, H_1, H_2, \ldots\}$ and $D^1 = [0, 2\pi]$. Let $g : D^1 \to G$ be such that $g(0) = e$ and $g(2\pi) \in H_m$ for some $m \in \{0, 1, 2, \ldots\}$. If we choose $s_0 = 0$ then $[g] \in \pi_1(G, H, e)$. The isomorphism $\gamma_1 : \pi_1(G, H, e) \to \pi_0(H, e)$ is given by $[g] \mapsto [g(2\pi)]$, where $[g(2\pi)] \equiv H_m$.

We use cylindrical coordinates (r, z, φ) as the coordinates of physical space and assume that the vortex is located on the z-axis. We define $A = \mathbb{R}^3 \backslash \mathbb{R}\mathbf{e}_z$, where $\mathbb{R}\,\mathbf{e}_z$ denotes the z-axis. We define $f : A \to G$ such that $f(r, z, 0) = e$ and $f(r, z, 2\pi) \in H_m$. Then for each fixed $(r, z)\, [f] \in \pi_1(G, H, e)$, $[pf] \in \pi_1(G/H, H)$ and the image of $[f]$ in the isomorphism γ_1 is $[f(r, z, 2\pi)] = H_m \in \pi_0(H, e)$. Thus f gives a vortex with the winding number represented by $H_m \in \pi_0(H, e)$. The corresponding order-parameter $x : A \to M$ is defined by $x(r, z, \varphi) = f(r, z, \varphi) \cdot x_{\text{ref}}$. For a vortex constructed this way $x(r, z, 0) = x_{\text{ref}}$ for all $r > 0, z \in \mathbb{R}$. This requirement can be relaxed, but for our purposes that is not necessary.

3. Spinor Bose–Einstein condensates

In this section we review the ground-state order-parameter spaces of spinor BECs. As a new result we show that the order-parameter space of the cyclic phase consists of two disconnected sets. A spinor in one set cannot be continuously converted into a spinor in the other set while staying in the order-parameter space all the time. Only the order-parameter space of the cyclic phase has this structure, since in other ground states the order-parameter space is connected.

A spinor condensate of atoms with hyperfine spin equal to F, $F = 1, 2, \ldots$, is described in the mean-field theory by the order-parameter ψ, which can be written in the form $\psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}\xi(\mathbf{r})$, where $n(\mathbf{r})$ is the particle density, $\xi(\mathbf{r})$ is the transpose of the complex vector $(\xi_F(\mathbf{r}), \xi_{F-1}(\mathbf{r}), \ldots, \xi_{-F}(\mathbf{r}))$ and $\xi(\mathbf{r})^\dagger\xi(\mathbf{r}) = 1$ [4, 18]. In the rest of the paper we call ξ the spinor. In some publications the term spinor refers to ψ, so one must be careful with the terminology. When we determine the order-parameter space, we set the density equal to one, so the order-parameter is just the spinor ξ.[1] If the normalization of ξ is the only restriction imposed on the order-parameter, the order-parameter space becomes S^{4F+1}. This space allows for no topological defects characterized by π_n with $n = 0, 1, 2, 3$. This is because from the theory of homotopy groups it is known that $\pi_n(S^{4F+1}) = \{e\}$ for $n = 0, 1, 2, 3$ and $F = 1, 2, 3, \ldots$. However, for example the order-parameter space of an $F = 1$ condensate may be only a subset of S^5. This can be inferred from the energy functional, which for an $F = 1$ condensate in the absence of external magnetic field reads [4, 18]

$$E[\psi] = \int \mathrm{d}^3\mathbf{r} \left\{ \frac{\hbar^2}{2M} \sum_{i=-1}^{+1} \nabla\psi_i^*(\mathbf{r}) \cdot \nabla\psi_i(\mathbf{r}) + V(\mathbf{r})n(\mathbf{r}) + \frac{n(\mathbf{r})^2}{2}[\alpha_1 + \beta_1\langle\mathbf{F}\rangle^2] \right\}. \tag{2}$$

Here $\mathbf{F}$ is the (hyperfine) spin operator, $\langle\mathbf{F}\rangle = \xi^\dagger(\mathbf{r})\mathbf{F}\xi(\mathbf{r})$, M is the mass of the atom, V is the external potential, $\alpha_1 = \frac{4\pi\hbar^2}{M}\frac{a_0+2a_2}{3}$, $\beta_1 = \frac{4\pi\hbar^2}{M}\frac{a_2-a_0}{3}$, and a_F is the s-wave scattering length in the total spin F channel.

One sees that it is energetically favoured that either $|\langle\mathbf{F}\rangle| = 0$ or $|\langle\mathbf{F}\rangle| = 1$, corresponding to the cases $\beta_1 > 0$ and $\beta_1 < 0$, respectively. When $\beta_1 > 0$ the system is said to be antiferromagnetic, whereas if $\beta_1 < 0$ the system is said to be ferromagnetic. It turns out that a group that acts transitively on the set of spinors fulfilling the condition $|\langle\mathbf{F}\rangle| = 1$ is $SU(2)$. This can be shown using a similar reasoning as that shown below in the context of the ground-state phases of $F = 2$ condensates. $SU(2)$ acts via its irreducible three-dimensional representation, i.e. the spin rotations of an $F = 1$-particle. For spinors with $|\langle\mathbf{F}\rangle| = 0$ we choose $G = \mathbb{R} \times SU(2)$, where $\mathbb{R}$ gives the gauge transformations of the spinor as $\xi \mapsto \mathrm{e}^{\mathrm{i}\theta}\xi, \theta \in \mathbb{R}$. As before, $SU(2)$ describes spin rotations.

For an $F = 2$ condensate the energy functional is [19]

$$E[\psi] = \int \mathrm{d}^3\mathbf{r} \left\{ \frac{\hbar^2}{2M} \sum_{i=-2}^{2} \nabla\psi_i^*(\mathbf{r}) \cdot \nabla\psi_i(\mathbf{r}) + V(\mathbf{r})n(\mathbf{r}) + \frac{n(\mathbf{r})^2}{2}(\alpha_2 + \beta_2\langle\mathbf{F}\rangle^2 + \gamma_2|\Theta(\mathbf{r})|^2) \right\}. \tag{3}$$

Here $\alpha_2 = \frac{1}{7}(4g_2 + 3g_4)$, $\beta_2 = -\frac{1}{7}(g_2 - g_4)$ and $\gamma_2 = \frac{1}{5}(g_0 - g_4) - \frac{2}{7}(g_2 - g_4)$, where $g_i = \frac{4\pi\hbar^2 a_i}{M}$ and $\Theta = 2\xi_2\xi_{-2} - 2\xi_1\xi_{-1} + \xi_0^2$. As in an $F = 1$ system, the energy is invariant in position-independent spin rotations and gauge transformations. The possible ground states have been calculated in [19, 20], and can be classified as follows. (i) If $\beta_2, \gamma_2 > 0$ the energy

[1] In principle the order-parameter space should be written as $M' = (\mathbb{R}_+ \times M) \cup \{0\}$, where $\mathbb{R}_+$ is the set of real numbers larger than zero giving the possible values of the square root of the density, and M gives the order-parameter space related to the density-independent part of the order-parameter. Now M consists of the possible values of the spinor, while in the case of a single component condensate $M = S^1$, which characterizes the possible values of the phase of the order-parameter. The point $\{0\}$ denotes the case where the density is zero. One sees that if M' is the order-parameter space there are no topologically stable defects. Any order-parameter can be converted into any other order-parameter via deformations which reduce the density to zero in an appropriate region of the physical space. From a physical point of view this is unlikely to happen, since reducing density to zero is not energetically favourable. In principle the necessary energy could come for example from thermal excitations, but in practice this is unlikely to happen. Thus one can ignore the zero of density. Then the order-parameter space becomes $\mathbb{R}_+ \times M$. For this $\pi_n(\mathbb{R}_+ \times M) = \pi_n(\mathbb{R}_+) \times \pi_n(M) = \pi_n(M)$, since $\pi_n(\mathbb{R}_+) = \{e\}$. Thus, from the point of view of topological defects, it is enough to study the structure of M only.

is minimized when $|\langle \mathbf{F} \rangle| = \Theta = 0$. Spinors with these properties are called cyclic. (ii) When $\beta_2 < 0, \gamma_2 > 0$ the minimum is obtained by making $|\langle \mathbf{F} \rangle| = 2, \Theta = 0$, and the ground state is ferromagnetic. (iii) If $\beta_2 > 0, \gamma_2 < 0$ the minimum is achieved by maximizing Θ, i.e. $|\Theta| = 1$, and $|\langle \mathbf{F} \rangle| = 0$. The ground state is called polar or antiferromagnetic. (iv) Finally, if α_2 and β_2 are both negative, the ground state is ferromagnetic for $4|\beta_2| > |\gamma_2|$, and polar otherwise. This is because $|\langle \mathbf{F} \rangle|$ and $|\Theta|$ cannot be maximized simultaneously.

Next we discuss briefly the structures of the order-parameter spaces of the ground states. Since the interaction energy is invariant in gauge transformations and spin rotations and β_2, γ_2 are arbitrary, also $|\Theta|$ and $\langle \mathbf{F} \rangle^2$ are invariant in gauge transformations and spin rotations. Furthermore, for every spinor ξ there is always a spin rotation R which rotates the spinor so that in the rotated state $R\xi$ the spin is parallel to the z-axis, i.e. $\langle \mathbf{F} \rangle = (R\xi)^\dagger \mathbf{F} R\xi = (R\xi)^\dagger F_z R\xi\, \mathbf{e}_z = \langle F_z \rangle\, \mathbf{e}_z$. In the rotated state $\langle F_x \rangle = \langle F_y \rangle = 0$. Thus the spin-dependent terms in equation (3) can be written as $\beta_2 \langle F_z \rangle^2 + \gamma_2 |\Theta|^2$. In the cyclic phase the energy is minimized when $\langle F_z \rangle = |\Theta| = 0$. Numerical calculations show that the solutions to these equations are (up to a gauge transformation and a rotation about the z-axis) $C0 = \frac{1}{2}(1, 0, \sqrt{2}, 0, -1)^T$, $C1 = \frac{1}{\sqrt{3}}(1, 0, 0, \sqrt{2}, 0)^T$ and $C1' = \frac{1}{\sqrt{3}}(0, \sqrt{2}, 0, 0, 1)^T$ [21, 22]. All cyclic spinors can be obtained from these spinors by a gauge transformation and a spin rotation. By exploiting the rotation matrix shown in the appendix one can show that $C1, C1'$ can be rotated into each other, while for $C0, C1$ and $C0, C1'$ this is not possible. This means that the order-parameter space of the cyclic phase consists of two disconnected sets, a fact that has not been pointed out before. Formally this can is expressed as $\pi_0(G/H) = \mathbb{Z}_2$. The group $G = \mathbb{R} \times SU(2)$ acts transitively on both disconnected set. In the ground state, the system can consist of regions, some of which are in a $C0$ state and others are in a $C1$ state. These regions are separated by a domain wall. This kind of structure is not possible in other zero-field ground-state phases of $F = 1$ or $F = 2$ condensates.

In the case of ferromagnetic ground states, the situation is simpler. Now the equations to be solved are $|\Theta| = \langle F_x \rangle = \langle F_y \rangle = 0$ and $|\langle F_z \rangle| = 2$. The only solutions (up to a phase) are $|F = 2, m_{\mathrm{F}} = 2\rangle$ and $|F = 2, m_{\mathrm{F}} = -2\rangle$. These spinors can be rotated into each other, so the order-parameter space is now connected. By examining the spinors obtained by a spin rotation from the reference order-parameters one sees that in the case of a ferromagnetic condensate we can choose $G = SU(2)$ instead of $\mathbb{R} \times SU(2)$.

Similar study for the polar phase shows that the order-parameter space of the polar phase is connected but larger than $\mathbb{R} \times SU(2)$. This complicates the study of polar defects, and they will not be discussed here [3].

Before going to the details of defects we review the possible ground-state phases of some alkali atom condensates. Experimental and theoretical results indicate that the $F = 1$ spinor Bose–Einstein condensate of ^{87}Rb is ferromagnetic [4, 6, 8, 23]. On the other hand, the $F = 2$ condensate of ^{87}Rb is probably polar [6, 7, 23]. The ^{85}Rb $F = 2$ condensate seems to be polar [23] and the ^{83}Rb isotope with $F = 2$ ferromagnetic [19].

^{23}Na scattering lengths determined in [24] indicate that ^{23}Na $F = 1$ is antiferromagnetic, as has been predicted in [4] and seen in experiments [5]. The ground-state phase of ^{23}Na $F = 2$ spinor condensate appears also to be antiferromagnetic [19]. Experimental study of this condensate is difficult because the $|F = 2, m_{\mathrm{F}} = 0\rangle$ state decays within milliseconds [25].

4. Examples of defects in spinor condensates

Next we present several examples of vortices and monopoles in spinor condensates. In these examples cylindrical coordinates (r, z, φ) and spherical coordinates (r, θ, φ) are used when discussing vortices and monopoles, respectively. In vortices the coordinate dependence is such that at $\varphi = 0$ the reference spinor is obtained. The vortex cores are assumed to be straight and located on the z-axis. Monopoles are located at the origin of the spherical coordinates. In what follows the minimum energy of a vortex with given winding number is also studied. This is calculated under the assumptions that the vortex stays fixed at the z-axis and the density n is independent of φ. The ground-state density can be assumed to cylindrically symmetric if the external potential used to confine the condensate is cylindrically symmetric and the vortex does not split. The latter is not true in general, since usually if a condensate is trapped in a harmonic trap it is energetically favourable for a vortex with winding number larger than one to split into winding number one vortices. In this case the particle density cannot be cylindrically symmetric. The splitting can possibly be prevented by using a trap which is steeper than harmonic [26] or by applying a repulsive potential in the vicinity of the rotation axis [27, p 245].

4.1. Ferromagnetic condensates

Next we briefly examine the vortex configurations of ferromagnetic condensates. Here we mean by ferromagnetic condensates systems whose ground-state spinor is $|F, m_{\mathrm{F}} = F\rangle$ or any other spinor obtained from this by a spin rotation. A separate gauge transformation is now unnecessary, as a spin rotation alone is able to produce that. We choose $|F, m_{\mathrm{F}} = F\rangle$ as our reference order-parameter. The isotropy group is isomorphic with $\mathbb{Z}_{2\mathrm{F}}$ and the order-parameter space is $SU(2)/\mathbb{Z}_{2\mathrm{F}}$; see [28]. The isotropy group for $F = 1$ and $F = 2$ has been calculated explicitly in [3]. The winding numbers of topologically stable vortices range from 1 to $2F - 1$. If the winding number of a vortex is m, the winding number of the antivortex is $2F - m$. Especially, a vortex with winding number F is also its antivortex. If $F = 1$, the order-parameter space is $SO(3)$, since $SU(2)/\mathbb{Z}_2 = SO(3)$ [4]. Because $\pi_2(SU(2)/\mathbb{Z}_{2\mathrm{F}}) = \{e\}$, in ferromagnetic condensates monopoles are not topologically stable. Next we construct order-parameters describing vortices in ferromagnetic $F = 1$ and $F = 2$ condensates. The rotation matrices needed in the calculations can be found in appendix A.

4.1.1. Ferromagnetic $F = 1$ condensate. Now $H = \{\mathbb{I}, -\mathbb{I}\} \in SU(2)$ and, as explained in section 2, a vortex is given by a map $f : \mathbb{R}^3\backslash\mathbb{R}\mathbf{e}_z \to SU(2)$ such that $f(r, z, 0) = \mathbb{I}$ and $f(r, z, 2\pi) = -\mathbb{I}$ for every $z \in \mathbb{R}, r > 0$. From the V-matrix (A.4) one sees that this kind of map is obtained by choosing τ such that $\tau(r, z, 0) = 0$ and $\tau(r, z, 2\pi) = 2\pi$. The functions α and β can be arbitrary functions of position. The corresponding order-parameter is $\psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}\xi(\mathbf{r})$, where $\xi(\mathbf{r}) = f(\mathbf{r}) \cdot \xi_{\mathrm{ref}} = V^{(1)}(\mathbf{r})\xi_{\mathrm{ref}}$ and $V^{(1)}$ is given in equation (A.6). We get

$$\psi(r, z, \varphi) = \sqrt{n(r, z, \varphi)} \begin{pmatrix} \left(\cos\frac{\tau}{2} - \mathrm{i}\cos\beta\sin\frac{\tau}{2}\right)^2 \\ -\sqrt{2}\,\mathrm{e}^{\mathrm{i}\alpha}\sin\beta\sin\frac{\tau}{2}\left(\mathrm{i}\cos\frac{\tau}{2} + \cos\beta\sin\frac{\tau}{2}\right) \\ -\mathrm{e}^{2\mathrm{i}\alpha}\sin^2\frac{\tau}{2}\sin^2\beta \end{pmatrix}. \tag{4}$$

From the way this was derived it follows that this is defined only for $r > 0$. This order-parameter can, however, be extended to whole $\mathbb{R}^3$ by requiring that $n(0, z, \varphi) = 0$. This is needed in order to keep the order-parameter well defined at $r = 0$.

Next we try to find a spinor giving a vortex with the minimum energy. Now the superfluid velocity is $\mathbf{v} = -\mathrm{i}\frac{\hbar}{M}\xi^\dagger\nabla\xi$. When the vortex has the smallest possible energy the velocity in radial and z-directions can be assumed to vanish and thus the spinor is a function of φ only. Additionally, the density is taken to be cylindrically symmetric. Using these assumptions the

Euler–Lagrange equations obtained from the energy functional show that the vortex energy is minimized when $\alpha = \text{constant}$, $\beta = \frac{\pi}{2}$ and $\tau(r, z, \varphi) = \varphi$. Thus the vortex takes the form

$$\psi(r, z, \varphi) = \sqrt{n(r, z)} \begin{pmatrix} \left(\cos\frac{\varphi}{2} - \mathrm{i}\cos\beta\sin\frac{\varphi}{2}\right)^2 \\ -\sqrt{2}\,\mathrm{e}^{\mathrm{i}\alpha}\sin\beta\sin\frac{\varphi}{2}\left(\mathrm{i}\cos\frac{\varphi}{2} + \cos\beta\sin\frac{\varphi}{2}\right) \\ -\mathrm{e}^{2\mathrm{i}\alpha}\sin^2\frac{\varphi}{2}\sin^2\beta \end{pmatrix}. \tag{5}$$

Here we have not set $\beta = \frac{\pi}{2}$, but have allowed it to have any value. This recalls a little the expression for the coreless vortex shown in [4]. These are, however, different defects, since the vortex of equation (5) cannot be converted into a non-vortex state by continuous deformations, whereas for the coreless vortex this can be done[2]. Another difference is that in the case a of coreless vortex the particle density does not vanish, while in the case of (5) the density vanishes on the z-axis. The superfluid velocity for the vortex state (5) is $\mathbf{v} = -\frac{\hbar}{M}\frac{\cos\beta}{r}\mathbf{e}_\varphi$ and if $\beta = \frac{\pi}{2}$ it vanishes. Thus, in contrast to a single-component condensate, in a spinor condensate the existence of a vortex does not have to lead to a nonzero superfluid velocity. The same phenomenon can be seen also in the orbital angular momentum. For the order-parameter (5) it is $\mathbf{L} = -N\hbar\cos\beta\mathbf{e}_z$, which also vanishes if $\beta = \frac{\pi}{2}$. The condensate does not have to contain any orbital angular momentum although there is a vortex in it. If the condensate is in a state with $\beta = 0$, it contains one unit of angular momentum per particle and the system is similar to a single-component condensate with a vortex. In the presence of dissipation the angular momentum does not have to be conserved, and the vortex can evolve towards the ground state where the angular momentum vanishes. The kinetic energy of (5) is proportional to $3 + \cos 2\beta$, which decreases monotonically as β increases from zero to $\frac{\pi}{2}$. Thus there is no energetic barrier which could render the vortex state with $\beta = 0$ metastable against conversion into the ground-state vortex.

A vortex obtained from (5) by setting $\beta = 0$ has been presented before in [4]. The general expression for the vortex shown in equation (5) or the behaviour of the angular momentum and superfluid velocity have, however, not been discussed before. This also holds true of the ferromagnetic $F = 2$ vortices discussed next.

4.1.2. Ferromagnetic $F = 2$ condensate. The isotropy group is $H = \{\mathbb{I}, -\mathrm{i}\sigma_z, (-\mathrm{i}\sigma_z)^2, (-\mathrm{i}\sigma_z)^3\}$, where σ_z is the z-component of the Pauli matrices. A map $f_m : \mathbb{R}^3\backslash\mathbb{R}\,\mathbf{e}_z \to SU(2)$ represents a vortex with winding number m if $f_m(r, z, 0) = \mathbb{I}$ and $f_m(r, z, 2\pi) = (-\mathrm{i}\sigma_z)^m$ for every $z \in \mathbb{R}, r > 0$. If we parametrize $SU(2)$ matrices as in (A.4), these conditions are fulfilled if $\tau(r, z, 0) = 0$ and $\tau(r, z, 2\pi) = m\pi$. Additionally, for $m = 1, 3$ the condition $\beta(r, z, 2\pi) = 2k\pi$ has to hold. Here k is an arbitrary integer. For $m = 2$ the function β can be arbitrary. The order-parameter becomes

$$\psi(r, z, \varphi) = \sqrt{n(r, z, \varphi)} \begin{pmatrix} \left(\cos\frac{\tau}{2} - \mathrm{i}\cos\beta\sin\frac{\tau}{2}\right)^4 \\ 2\,\mathrm{e}^{\mathrm{i}\alpha}\sin\frac{\tau}{2}\sin\beta\left(\mathrm{i}\cos\frac{\tau}{2} + \cos\beta\sin\frac{\tau}{2}\right)^3 \\ \sqrt{\frac{3}{8}}\,\mathrm{e}^{2\mathrm{i}\alpha}\sin^2\beta\left(\cos\beta - \cos\tau\cos\beta + \mathrm{i}\sin\tau\right)^2 \\ 2\,\mathrm{e}^{3\mathrm{i}\alpha}\sin^3\frac{\tau}{2}\sin^3\beta\left(\mathrm{i}\cos\frac{\tau}{2} + \sin\frac{\tau}{2}\cos\beta\right) \\ \mathrm{e}^{4\mathrm{i}\alpha}\sin^4\frac{\tau}{2}\sin^4\beta \end{pmatrix}. \tag{6}$$

The minimum energy of a vortex with winding number m is obtained when $\tau(r, z, \varphi) = \frac{m\varphi}{2}$ and $\xi = \xi(\varphi)$. If $m = 2$ the vortex minimizing the energy is obtained from equation (6)

[2] The conservation of magnetization may change the situation. This is discussed in section 5.

by replacing τ with φ and setting $\beta = \frac{\pi}{2}$. In this state, the superfluid velocity and angular momentum vanish. We show that this is a general property of a winding number F vortex in a ferromagnetic condensate with hyperfine spin F. This vortex is represented by the element $-\mathbb{I} \in H$ and a vortex can be written as $\psi(r, z, \varphi) = \sqrt{n(r, z)}\xi(\varphi)$, where $\xi(\varphi) = \mathrm{e}^{-\mathrm{i}\varphi\mathbf{n}\cdot\mathbf{F}}\xi_{\mathrm{ref}}$, $\mathbf{n} = (\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta)$ is a constant vector and $\xi_{\mathrm{ref}} = |F, m_{\mathrm{F}} = F\rangle$. If $\beta = \frac{\pi}{2}$ we get $\mathbf{n} = (\cos\alpha, \sin\alpha, 0)$ and $v_\varphi, L_z \sim \xi^\dagger \frac{\partial}{\partial\varphi}\xi = -\mathrm{i}\xi_{\mathrm{ref}}\,\mathrm{e}^{\mathrm{i}\varphi\mathbf{n}\cdot\mathbf{F}}\mathbf{n}\cdot\mathbf{F}\,\mathrm{e}^{-\mathrm{i}\varphi\mathbf{n}\cdot\mathbf{F}}\xi_{\mathrm{ref}} = -\mathrm{i}\xi_{\mathrm{ref}}\mathbf{n}\cdot\mathbf{F}\xi_{\mathrm{ref}} = 0$.

If $m = 1, 3$ the function β that minimizes the energy of a vortex is more difficult to find. The obvious choice $\beta \equiv 0$ is not the correct one, since for example $\beta(\varphi) = \frac{8\pi+\varphi}{5}$ produces a smaller energy. Finding the ground states of vortices with $m = 1, 3$ will be left for future publications, and will not be discussed here in more detail. If $\beta = 0$ the vortex is simply $\psi(r, z, \varphi) = \sqrt{n(r, z)}\,\mathrm{e}^{\mathrm{i}m\varphi}|F = 2, m_{\mathrm{F}} = 2\rangle$, $m = 1, 3$.

4.2. *Antiferromagnetic $F = 1$ condensate*

It is advantageous to use the U matrix of equation (A.3) and corresponding representation matrices when discussing the defects of an antiferromagnetic $F = 1$ condensate. The reference order-parameter is chosen to be $|F = 1, m_{\mathrm{F}} = 0\rangle$ and the isotropy group is $H = \left\{(m2\pi, U(\varphi, 0, 0)), \left(\left(m + \frac{1}{2}\right)2\pi, gU(\varphi, 0, 0)\right) \middle| \varphi \in [0, 4\pi], m \in \mathbb{Z}\right\}$, where we have defined $g = U(0, \pi, 0)$. The connected component of the identity is $H_0 = \{(0, U(\varphi, 0, \varphi)) | \varphi \in [0, 2\pi]\}$ [3]. Now vortices and monopoles are possible and both of them are classified by integers. A general antiferromagnetic spinor can then be written as $\psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}\xi(\mathbf{r})$, where $\xi(\mathbf{r}) = \mathrm{e}^{\mathrm{i}\theta(\mathbf{r})}U^{(1)}(\mathbf{r})\xi_{\mathrm{ref}}$ and $U^{(1)}$ is shown in (A.5). Explicitly

$$\psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}\,\mathrm{e}^{\mathrm{i}\theta(\mathbf{r})} \begin{pmatrix} -\mathrm{e}^{-\mathrm{i}\alpha(\mathbf{r})}\frac{1}{\sqrt{2}}\sin\beta(\mathbf{r}) \\ \cos\beta(\mathbf{r}) \\ \mathrm{e}^{\mathrm{i}\alpha(\mathbf{r})}\frac{1}{\sqrt{2}}\sin\beta(\mathbf{r}) \end{pmatrix}. \tag{7}$$

4.2.1. *Vortices.*

As can be seen from the isotropy group H, vortices with winding numbers m and $m + \frac{1}{2}$ are possible ($m \in \mathbb{Z}$). Using an analysis similar to that of ferromagnetic condensates, a vortex with a winding number m can be shown to be given by (7) if θ fulfils the condition $\theta(r, z, 0) = 0$, $\theta(r, z, 2\pi) = 2\pi m$. Other angles can be chosen freely as long as $\xi(r, z, 0) = \xi(r, z, 2\pi) = |F = 1, m_{\mathrm{F}} = 0\rangle$. For $m + \frac{1}{2}$ condensates the requirements are $\theta(r, z, 0) = 0$, $\theta(r, z, 2\pi) = 2\pi\left(m + \frac{1}{2}\right)$ and $\xi(r, z, 0) = -\xi(r, z, 2\pi) = |F = 1, m_{\mathrm{F}} = 0\rangle$. The superfluid velocity is $\mathbf{v} = \frac{\hbar}{Mr}\frac{\partial\theta}{\partial\varphi}\mathbf{e}_\varphi$ and it cannot vanish everywhere if there is a vortex in the system. The orbital angular momentum is $\mathbf{L} = N\hbar l\mathbf{e}_z$, where l is either m or $m + \frac{1}{2}$. In the following we assume that $\psi(r, z, \varphi) = \sqrt{n(r, z)}\xi(\varphi)$. The spinor minimizing the energy of a vortex with winding number $m \in \mathbb{Z}$ is obtained when α, β are constants and $\theta(\varphi) = m\varphi$. If we choose $\beta = 0$ it becomes $\psi_m(r, z, \varphi) = \sqrt{n(r, z)}\,\mathrm{e}^{\mathrm{i}m\varphi}|F = 1, m_{\mathrm{F}} = 0\rangle$. A winding number $m + \frac{1}{2}$ vortex with minimum energy is given by (7) under the conditions that $\theta(r, z, 2\pi) = \left(2m + \frac{1}{2}\right)\pi$, α is constant and $\beta = \frac{\varphi}{2}$,

$$\psi_{m+\frac{1}{2}}(r, z, \varphi) = \sqrt{n(r, z)}\,\mathrm{e}^{\mathrm{i}(m+\frac{1}{2})\varphi} \begin{pmatrix} -\mathrm{e}^{-\mathrm{i}\alpha}\frac{1}{\sqrt{2}}\sin\frac{\varphi}{2} \\ \cos\frac{\varphi}{2} \\ \mathrm{e}^{\mathrm{i}\alpha}\frac{1}{\sqrt{2}}\sin\frac{\varphi}{2} \end{pmatrix}. \tag{8}$$

Properties of vortices of type (8) have been discussed before for example in [29–32].

4.2.2. Monopoles. As explained in section 2, an expression for a winding number m monopole can be found as an extension of a map $g_m : [0, 2\pi] \to H_0$, where g_m determines the element of $\pi_1(H_0, e)$ with winding number m. Now we define $g_m(\varphi) = (0, U(\alpha_m(\varphi), 0, \alpha_m(\varphi)))$, where α_m is such that $\alpha_m(0) = 0, \alpha_m(2\pi) = 2\pi m$. We define a map $\tilde{g}_m : D^2 \to \mathbb{R} \times SU(2)$ such that $\tilde{g}_m(\pi, \varphi) = g_m(\varphi)$. Then $f_m : \mathbb{R}^3\backslash\{\mathbf{0}\} \to [\mathbb{R} \times SU(2)]/H$, $f_m(r, \theta, \varphi) = \tilde{g}_m(\theta, \varphi)H$ gives a monopole with winding number m. A general form for $\tilde{g}_m$ is $\tilde{g}_m(\theta, \varphi) = (\delta_m(\theta, \varphi), U(\alpha_m(\theta, \varphi), \beta_m(\theta, \varphi), \alpha_m(\theta, \varphi)))$ with the conditions $\alpha_m(\pi, \varphi) = \alpha_m(\varphi), \beta_m(\pi, \varphi) = \delta_m(\pi, \varphi) = 0$, which follow from $\tilde{g}_m(\theta = \pi, \varphi) = g_m(\varphi)$. From the continuity of $\tilde{g}_m$ it also follows that $\delta_m(0, \varphi) = 0$ and $\beta_m(0, \varphi) = (2k + 1)\pi$ with k an arbitrary integer.

By a simple change of basis equation (7) can be cast in the form

$$\psi_m(r, \varphi, \theta) = \sqrt{n(r, \theta, \varphi)}\, \mathrm{e}^{\mathrm{i}\delta} \begin{pmatrix} \cos\alpha_m \sin\beta_k \\ \sin\alpha_m \sin\beta_k \\ -\cos\beta_k \end{pmatrix}. \tag{9}$$

Here we have substituted $\delta, \alpha_m, \beta_k$ for θ, α, β, respectively. If $\delta \equiv 0$ the above spinor is a real and normalized three-component vector. Then there is an integral equation

$$w = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \mathrm{d}\varphi\, \mathrm{d}\theta\, \xi \cdot \left(\frac{\partial \xi}{\partial \theta} \times \frac{\partial \xi}{\partial \varphi} \right) \tag{10}$$

which gives the winding number w of the monopole; see [12] or [33]. Using this equation one can confirm that by assuming $\delta_m \equiv 0$ and that g_m fulfils the conditions stated above, the winding number of the spinor in equation (9) is m. For a non-constant δ_m the winding number is the same, as it does not depend on the form of δ_m. From equation (9) one sees that the superfluid velocity vanishes if and only if δ_m is a constant function. This follows from the fact that $\delta \equiv \text{constant} \iff \xi^\dagger \nabla \xi \in \mathbb{R}^3$. Because $\mathbf{v} = -\mathrm{i}\frac{\hbar}{M}\xi^\dagger \nabla \xi \in \mathbb{R}^3$, it follows that $\mathbf{v} = \mathbf{0}$ if and only if δ_m is a constant.

While it is instructive to know what are the general requirements for a spinor to represent a monopole, it is important to see that the typical expression for a monopole can also be obtained as a special case of above equations. If we choose g_m such that $\delta_m \equiv 0, \alpha_m(\theta, \varphi) = m\varphi$ and $\beta_m(\theta, \varphi) = (2k + 1)(\pi - \theta)$, where k is an arbitrary integer, all the requirements for δ_m, α_m and β_m are fulfilled and the monopole becomes

$$\psi_m(r, \varphi, \theta) = \sqrt{n(r, \theta, \varphi)} \begin{pmatrix} \cos(m\varphi) \sin[(2k + 1)\theta] \\ \sin(m\varphi) \sin[(2k + 1)\theta] \\ \cos[(2k + 1)\theta] \end{pmatrix}. \tag{11}$$

Choosing $k = 0$ gives the usual expression for a monopole. It has been presented in the context of BEC before for example in [34, 35], but the requirements for a general expression of a monopole and its properties have not been discussed before. A monopole with $m = 1, k = 0$ has been numerically studied in [34, 35].

4.3. Cyclic states

The defects of the $C0$ phase have already been discussed in [3], so here we concentrate on the defects of the $C1$ phase. The isotropy group H has not been calculated before, so we have to do it here. For the $SU(2)$-matrices we now use the U-representation shown in (A.3). The reference spinor is chosen to be $C1 = \frac{1}{\sqrt{3}}(1, 0, 0, \sqrt{2}, 0)^T$ and a general expression for a spinor is

$$\psi(r,z,\varphi) = \sqrt{n(r,z,\varphi)}\,\mathrm{e}^{\mathrm{i}\theta}\begin{pmatrix} \frac{\mathrm{e}^{-2\mathrm{i}(\alpha+\gamma)}}{\sqrt{3}}\left[\cos^4\frac{\beta}{2} - \sqrt{2}\,\mathrm{e}^{3\mathrm{i}\gamma}\sin^2\frac{\beta}{2}\sin\beta\right] \\ \frac{\mathrm{e}^{-\mathrm{i}(\alpha+2\gamma)}}{\sqrt{6}}\left[\mathrm{e}^{3\mathrm{i}\gamma}(\cos\beta - \cos(2\beta)) + \sqrt{2}\cos^2\frac{\beta}{2}\sin\beta\right] \\ \frac{\mathrm{e}^{-2\mathrm{i}\gamma}}{4}\sin\beta[-4\,\mathrm{e}^{3\mathrm{i}\gamma}\cos\beta + \sqrt{2}\sin\beta] \\ \frac{\mathrm{e}^{\mathrm{i}(\alpha-2\gamma)}}{\sqrt{6}}\left[\mathrm{e}^{3\mathrm{i}\gamma}(\cos\beta + \cos(2\beta)) + \sqrt{2}\sin^2\frac{\beta}{2}\sin\beta\right] \\ \frac{\mathrm{e}^{2\mathrm{i}(\alpha-\gamma)}}{\sqrt{3}}\left[\sin^4\frac{\beta}{2} + \sqrt{2}\,\mathrm{e}^{3\mathrm{i}\gamma}\cos^2\frac{\beta}{2}\sin\beta\right] \end{pmatrix}. \tag{12}$$

Equating this with the reference spinor $C1$ shows that the isotropy group is $H = \left\{\left(\frac{2\pi}{3}(2n+3m), U\left(\frac{2\pi n}{3}, 0, 0\right)\right) \middle| m \in \mathbb{Z}, n = 0, \ldots, 5\right\} \subset \mathbb{R} \times SU(2)$ and $H_0 = (0, \mathbb{I})$. This means that only vortices, not monopoles, are topologically stable. $H/H_0 = H$ is isomorphic to the group $\mathbb{Z}\times\mathbb{Z}_6$, the isomorphism is given by the map $\left(\frac{2\pi}{3}(2n+3m), U\left(\frac{2\pi n}{3}, 0, 0\right)\right) \mapsto (m,n) \in \mathbb{Z}\times\mathbb{Z}_6$. Therefore vortices are classified by two winding numbers (m,n) and those with different m cannot be converted into one another. However, for vortices with winding numbers (m,n) and $(m, n+6)$ this is possible.

A vortex with winding numbers (m,n) is given by $f_{m,n} : \mathbb{R}^3\backslash\mathbb{R}\,\mathbf{e}_z \to \mathbb{R} \times SU(2)$ such that $f_{m,n}(r,z,0) = e$ and $f_{m,n}(r,z,2\pi) = \left(\frac{2\pi}{3}(2n+3m), U\left(\frac{2\pi n}{3}, 0, 0\right)\right)$ for all $z \in \mathbb{R}, r > 0$. Now we define $f_{m,n}$ by $f_{m,n}(r,z,\varphi) = \left(\frac{\varphi}{3}(2n+3m), U\left(\frac{\varphi n}{3}, 0, 0\right)\right)$. The corresponding order-parameter is

$$\psi(r,z,\varphi) = \sqrt{n(r,z)}\begin{pmatrix} \frac{1}{\sqrt{3}}\,\mathrm{e}^{\mathrm{i}\varphi m} \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}}\,\mathrm{e}^{\mathrm{i}\varphi(m+n)} \\ 0 \end{pmatrix}. \tag{13}$$

This is also the order-parameter that minimizes the vortex energy. The superfluid velocity and angular momentum are $\mathbf{v} = -\frac{\hbar}{Mr}\left(m + \frac{2n}{3}\right)\mathbf{e}_\varphi$ and $\mathbf{L} = N\hbar\left(m + \frac{2n}{3}\right)\mathbf{e}_z$.

5. Effects of external magnetic field and magnetization

So far we have assumed that there is no external magnetic field present. Now we consider the situation where the condensate is placed in a magnetic field directed parallel to the z-axis. To the second order in the strength B of the magnetic field the energy from the field can be written as [27]

$$E_B[\psi] = \int \mathrm{d}^3r\, n(\mathbf{r})\left[\gamma B(\mathbf{r})\langle F_z\rangle + \epsilon B^2(\mathbf{r})\left\langle F_z^2\right\rangle\right], \tag{14}$$

where γ and ϵ are constants. For ^{87}Rb and ^{23}Na these are $\gamma = \pm\frac{\mu_\mathrm{B}}{2}$ and $\epsilon = \mp\frac{\mu_\mathrm{B}^2}{4\Delta E_\mathrm{hf}}$. Here μ_B is the Bohr magneton, ΔE_hf is the hyperfine splitting between $F = 2$ and $F = 1$ states and upper (lower) sign refers to $F = 2$ ($F = 1$). In the presence of a magnetic field, the conservation of magnetization has to be taken into attention. If the magnetic field is parallel to the z-axis the magnetization is defined by $M = \int \mathrm{d}^3r\, n(\mathbf{r})\langle F_z\rangle = \int \mathrm{d}^3r\, n(\mathbf{r}) \sum_{j=-F}^{F} j|\xi_j(\mathbf{r})|^2 (\hbar = 1)$. Magnetization can have any value between $-NF$ and NF, where N is the particle number. In the absence of an external magnetic field the magnetization is a meaningless quantity because of the lack of a well-defined quantization axis of the hyperfine spin. In the presence of a magnetic field the magnetization is a well-defined and often conserved quantity since the collisions that do not conserve magnetization occur usually in a timescale much longer than a typical lifetime of the condensate. The conservation of magnetization has been experimentally verified [6, 8]. Thus the energy of the system should be minimized under the assumption of fixed M. If we consider weak values of the magnetic field it is enough to include only the first term in equation (14) in the energy. If the magnetic field is spatially constant this term is just magnetization multiplied by a constant. Thus the ground state is determined by the value of magnetization, and does not depend on the strength of the magnetic field, as long as the quadratic term in the magnetic field is negligible.

If the external field is absent the minimum of energy is obtained (if $F = 1$) when either $|\langle \mathbf{F}\rangle| = 0$ or $|\langle \mathbf{F}\rangle| = 1$. If external field is present the energy has to be minimized under the assumption of conserved magnetization. For a ferromagnetic $F = 1$ condensate any magnetization can be produced while $|\langle \mathbf{F}\rangle| = 1$. Thus in this case the ground state is obtained if one finds an order-parameter which produces the given magnetization and fulfils the condition $|\langle \mathbf{F}\rangle| = 1$. On the other hand, if $|\langle \mathbf{F}\rangle| = 0$ the only possible value for magnetization is zero. If the magnetization is nonzero and the system is antiferromagnetic, the condition $|\langle \mathbf{F}\rangle| = 0$ cannot be satisfied everywhere. In this case the calculation of the ground state is more difficult. The situation is similar in the cyclic and polar (antiferromagnetic) phases of $F = 2$ system.

Next we assume that only the term linear in the magnetic field is included in the energy and show that the previously calculated vortex of a ferromagnetic $F = 1$ condensate is possible for an arbitrary value of the magnetization. Direct calculation shows that for the vortex of equation (5) the magnetization is $N\cos^2\beta$. Any magnetization between 0 and N can be obtained as β is varied between $\frac{\pi}{2}$ and 0. The vortex with minimum energy is obtained when magnetization vanishes. Because equation (5) is obtained using $|F = 1, m_{\rm F} = 1\rangle$ as the reference order-parameter, it is not surprising that positive values of magnetization are favoured by it. A vortex configuration which has negative magnetization can be obtained by choosing $|F = 1, m_{\rm F} = -1\rangle$ as the reference spinor and calculating a representative of a vortex as before. The magnetization turns out to be equal to $-N\cos^2\beta$. The vortices corresponding to different choices of the reference order-parameter have different magnetizations, but if the magnetization is allowed to vary, these vortices can be converted continuously into one another. Thus they are similar from topological point of view.

Next we see that setting magnetization fixed makes new kind of defects stable. If we choose $\beta = \frac{\pi}{2}$ in equation (4), multiply the spinor by $\mathrm{e}^{\mathrm{i}\theta}$ and redefine $\alpha \to \alpha + \frac{\pi}{2}$, we get

$$\psi(r, z, \varphi) = \sqrt{n(r, z, \varphi)} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta}\cos^2\frac{\tau}{2} \\ \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}(\alpha+\theta)}\sin\tau \\ \mathrm{e}^{\mathrm{i}(2\alpha+\theta)}\sin^2\frac{\tau}{2} \end{pmatrix}. \tag{15}$$

This is an alternative parametrization for a ferromagnetic spinor. From the form of the order-parameter one might assume that a vortex is obtained if $\theta = m\varphi$ and $\alpha = n\varphi$ with m, n integers. Like before, we assume that the vortices are straight and located on the z-axis. We also require that $n = n(r, z)$, $\tau = \tau(r, z)$. The latter assumptions mean that the particle density of each spin component is cylindrically symmetric[3]. The magnetization is $M = \int \mathrm{d}^3\mathbf{r}\, n(\mathbf{r})\cos\tau(r, z)$, which shows that (15) can produce any magnetization if τ is chosen properly. Topologically the decay of a vortex in one component can now only be achieved by converting all atoms from that spin state into other states. From the above spinor one sees that always two components vanish simultaneously. When this happens magnetization is $\pm N$. Thus if the magnetization is fixed and different from $\pm N$, one cannot make any spin component vanish while keeping the magnetization fixed. If the magnetization is different from $\pm N$, the order-parameter space is

[3] These kinds of vortices are called *axisymmetric* in [37, 39].

$S^1 \times S^1$, since the phase of two components can be chosen freely. The dynamical stability and dynamics of these type of vortices has been studied in [36–40]. However, their topological stability following from the conservation of magnetization has not been pointed out before. If the order-parameter is of the form (15), vortices appear in each spin component separately. This means that the total density does not have to vanish, unless there is a vortex in every component. If there is one spin component without vortex the system is a coreless vortex. For example by choosing $m = 0, n = 1$ or $m = 1, n = -1$ one gets an expression for a coreless vortex.

If the magnetization is not a conserved quantity, vortices with even m can be converted into a uniform configuration and those with odd m are equivalent with the winding number one vortex of equation (5). This is because in zero field τ can be converted into a map for which $\tau(\mathbf{r}) = 0$ holds for all $\mathbf{r}$. Then the spinor becomes $e^{im\varphi}|F = 1, m_F = 1\rangle$. This in turn is equivalent with a vortex with winding number zero or one for m even or odd, respectively; see [4].

Above we have assumed that the particle densities of different spin components are symmetric. It is possible that a deviation from this allows a continuous decay of vortex. This is also indicated by numerical studies [39].

If the magnetic field is strong, the term quadratic in the magnetic field has to be included in the energy. Also in this case the magnetization is conserved, and vortices which are possible when only the linear term is included remain topologically stable.

6. Creation of defects

Next we propose a method to create vortices in a ferromagnetic condensate. These vortices are topologically stable also in the absence of an external magnetic field. In [41, 42] a way to create a vortex exploiting the spin degree of freedom has been studied. In this method a condensate in a low-field seeking state is prepared in an Ioffe–Pritchard trap. Initially the magnetic field in the z-direction is assumed to be much stronger than the magnetic field $B_\perp$ in the xy-plane. The z-component of the field B_z is then reversed slowly, while keeping $B_\perp$ fixed. In this way a vortex with winding number $2F$ is created, F being the hyperfine spin of the condensate atoms. The feasibility of this method has been experimentally verified using a ^{23}Na condensate prepared in the low-field seeking states $|F = 1, m_F = -1\rangle$ or $|F = 2, m_F = +2\rangle$ [43]. Since a vortex with a winding number larger than one is energetically unstable against decay into winding number one vortices, it is presumed that vortices created this way will split. In the case of $F = 1$ ^{23}Na condensate this has been seen to occur [44], and it is expected to happen also in the $F = 2$ case. In addition to creating defects which are stable in the presence of magnetic field, we propose how a modification of this method can be used to create $2F$ vortices with winding number one in a ferromagnetic condensate with hyperfine spin F. These vortices are stable in the absence of an external magnetic field. To create these vortices, in addition to an Ioffe–Prichard trap, an optical trap is needed. First a vortex with winding number $2F$ is created in the previously described way. Then one waits until the vortex decays into winding number one vortices. Then $B_\perp$ is turned off, and the optical trap is turned on simultaneously. This does not change the spin state because it is assumed that $B_z \gg B_\perp$. The remaining field in the z-direction can then be reduced to zero, and vortices are allowed to evolve in the optical trap. If the magnetic trap is turned off before the vortex has split, the vortex can in principle continuously convert into a non-vortex state. This is possible, for example, if the vortex reverses the rotation which was used to create it. This is prevented by letting the vortex split before turning off the Ioffe–Pritchard trap. In experiments there are usually stray ac magnetic fields, so the creation of zero-field defects requires effective magnetic shielding, which can be troublesome. Achieving this would, however, be rewarding, as it would allow one to change the topological stability of vortices as a function of the magnetic field.

The creation of monopoles is not as straightforward as that of vortices, but a method for this has been proposed in [45].

7. Conclusions

In this paper we have first derived a systematic way to create explicit expressions for vortices and monopoles. This method requires the calculation of the first and second homotopy groups of the order-parameter space G/H. This can be achieved by using equations obtained from the exact sequence of relative homotopy groups. After this the expressions for defects can be constructed by finding suitable mappings from the physical space into the group G.

We have created examples of vortices and monopoles in spinor Bose–Einstein condensates using this method. Especially the defects in zero external magnetic field have been discussed. We have presented examples of vortices in ferromagnetic $F = 1$ and $F = 2$ condensates and vortices and monopoles in antiferromagnetic $F = 1$ condensate. We also pointed out that the order-parameter space of the cyclic phase of $F = 2$ condensate consists of two disconnected sets. The properties of one of the sets have been studied previously in [3]. Here we calculated the topological defects of the other set and showed that vortices classified by $\mathbb{Z} \times \mathbb{Z}_6$ are topologically stable, whereas monopoles are not possible. Also the superfluid velocity induced by the defects is examined. It has been shown that in a ferromagnetic condensate with hyperfine spin F the presence of a vortex with winding number F does not have to induce nonzero superfluid velocity or orbital angular momentum.

We have also studied the effect of a magnetic field, concentrating on a ferromagnetic $F = 1$ condensate. It has been shown that a vortex which is topologically stable in the absence of a magnetic field is also possible if the magnetization has a fixed value. This means that it is also possible if there is a homogeneous magnetic field present. In addition to this, we have found out that the conservation of magnetization may stabilize vortices which are not topologically stable if the magnetization can vary freely. Thus there can be a transition from a vortex state to a non-vortex state as the magnetic field strength is lowered to zero. Finally a method to create vortices which are topologically stable in the absence of a magnetic field has been suggested.

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Appendix. Spin matrices and rotation operators

The spin matrices for $F = 1$ are

$$F_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad F_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad F_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{A.1}$$

and those of $F = 2$ are

$$F_x = \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}, \qquad F_y = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix},$$
$$F_z = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}. \tag{A.2}$$

Here we have set $\hbar = 1$. Depending on the system studied, the elements of $SU(2)$ have been written either in the form

$$U(\alpha, \beta, \gamma) = \mathrm{e}^{-\mathrm{i}\alpha F_z}\,\mathrm{e}^{-\mathrm{i}\beta F_y}\,\mathrm{e}^{-\mathrm{i}\gamma F_z} = \begin{pmatrix} \cos\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}(\alpha+\gamma)/2} & -\sin\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}(\gamma-\alpha)/2} \\ \sin\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}(\gamma-\alpha)/2} & \cos\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}(\alpha+\gamma)/2} \end{pmatrix}, \tag{A.3}$$

or in the form

$$V(\tau, \alpha, \beta) = \mathrm{e}^{-\mathrm{i}\frac{\tau}{2}\mathbf{n}\cdot\sigma} = \begin{pmatrix} \cos\frac{\tau}{2} - \mathrm{i}\sin\frac{\tau}{2}\cos\beta & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha}\sin\frac{\tau}{2}\sin\beta \\ -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\alpha}\sin\frac{\tau}{2}\sin\beta & \cos\frac{\tau}{2} + \mathrm{i}\sin\frac{\tau}{2}\cos\beta \end{pmatrix}, \tag{A.4}$$

where $\mathbf{n} = (\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta)$ and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is a vector formed from Pauli matrices. The V-matrix has been used when discussing the ferromagnetic condensates, otherwise the U-matrix has been used. $(2F + 1)$-dimensional irreducible representations of these matrices are given by the maps $U(\alpha, \beta, \gamma) \mapsto U^{(F)}(\alpha, \beta, \gamma)$ and $V(\tau, \alpha, \beta) \mapsto V^{(F)}(\tau, \alpha, \beta)$. Here $U^{(F)}(\alpha, \beta, \gamma) = \exp(-\mathrm{i}\alpha F_z)\exp(-\mathrm{i}\beta F_y)\exp(-\mathrm{i}\gamma F_z)$, $V^{(F)}(\tau, \alpha, \beta) = \exp(-\mathrm{i}\tau\mathbf{n}\cdot\mathbf{F})$ and $\mathbf{F}$ is the spin operator of spin F system. In this paper, we need explicit expressions for $U^{(F)}$ and $V^{(F)}$ for $F = 1, 2$. For $F = 1$ these are

$$U^{(1)}(\alpha, \beta, \gamma) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}(\alpha+\gamma)}\cos^2\frac{\beta}{2} & -\mathrm{e}^{-\mathrm{i}\alpha}\frac{1}{\sqrt{2}}\sin\beta & \mathrm{e}^{-\mathrm{i}(\alpha-\gamma)}\sin^2\frac{\beta}{2} \\ \mathrm{e}^{-\mathrm{i}\gamma}\frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\mathrm{e}^{\mathrm{i}\gamma}\frac{1}{\sqrt{2}}\sin\beta \\ \mathrm{e}^{\mathrm{i}(\alpha-\gamma)}\sin^2\frac{\beta}{2} & \mathrm{e}^{\mathrm{i}\alpha}\frac{1}{\sqrt{2}}\sin\beta & \mathrm{e}^{\mathrm{i}(\alpha+\gamma)}\cos^2\frac{\beta}{2} \end{pmatrix} \tag{A.5}$$

and

$$V^{(1)}(\tau, \alpha, \beta) = \left(\begin{matrix} \left(\cos\frac{\tau}{2} - \mathrm{i}\cos\beta\sin\frac{\tau}{2}\right)^2 & \mathrm{e}^{-\mathrm{i}\alpha}\sin\beta\frac{(-1+\cos\tau)\cos\beta-\mathrm{i}\sin\tau}{\sqrt{2}} \\ -\sqrt{2}\,\mathrm{e}^{\mathrm{i}\alpha}\sin\beta\sin\frac{\tau}{2}\left(\mathrm{i}\cos\frac{\tau}{2} + \cos\beta\sin\frac{\tau}{2}\right) & \cos^2\frac{\tau}{2} + \cos(2\beta)\sin^2\frac{\tau}{2} \\ -\mathrm{e}^{2\mathrm{i}\alpha}\sin^2\frac{\tau}{2}\sin^2\beta & -\mathrm{e}^{\mathrm{i}\alpha}\sin\beta\frac{(-1+\cos\tau)\cos\beta+\mathrm{i}\sin\tau}{\sqrt{2}} \end{matrix}\right.$$
$$\left.\begin{matrix} -\mathrm{e}^{-\mathrm{i}2\alpha}\sin^2\frac{\tau}{2}\sin^2\beta \\ -\mathrm{e}^{-\mathrm{i}\alpha}\sin\beta\frac{(-1+\cos\tau)\cos\beta+\mathrm{i}\sin\tau}{\sqrt{2}} \\ \left(\cos\frac{\tau}{2} + \mathrm{i}\cos\beta\sin\frac{\tau}{2}\right)^2 \end{matrix}\right). \tag{A.6}$$

For $F = 2$ the spin rotation matrix is

$$U^{(2)}(\alpha,\beta,\gamma)=\begin{pmatrix} e^{-2i(\alpha+\gamma)}\cos^4\frac{\beta}{2} & -e^{-i(2\alpha+\gamma)}\sin\beta\cos^2\frac{\beta}{2} & e^{-2i\alpha}\frac{\sqrt{6}}{4}\sin^2\beta & -e^{-i(2\alpha-\gamma)}\sin\beta\sin^2\frac{\beta}{2} & e^{-2i(\alpha-\gamma)}\sin^4\frac{\beta}{2} \\ e^{-i(\alpha+2\gamma)}\sin\beta\cos^2\frac{\beta}{2} & e^{-i(\alpha+\gamma)}\frac{1}{2}(\cos\beta+\cos 2\beta) & -e^{-i\alpha}\frac{\sqrt{6}}{4}\sin 2\beta & e^{-i(\alpha-\gamma)}\frac{1}{2}(\cos\beta-\cos 2\beta) & -e^{-i(\alpha-2\gamma)}\sin\beta\sin^2\frac{\beta}{2} \\ e^{-2i\gamma}\frac{\sqrt{6}}{4}\sin^2\beta & e^{-i\gamma}\frac{\sqrt{6}}{4}\sin 2\beta & \frac{1}{4}(1+3\cos 2\beta) & -e^{i\gamma}\frac{\sqrt{6}}{4}\sin 2\beta & e^{2i\gamma}\frac{\sqrt{6}}{4}\sin^2\beta \\ e^{i(\alpha-2\gamma)}\sin\beta\sin^2\frac{\beta}{2} & e^{i(\alpha-\gamma)}\frac{1}{2}(\cos\beta-\cos 2\beta) & e^{i\alpha}\frac{\sqrt{6}}{4}\sin 2\beta & e^{i(\alpha+\gamma)}\frac{1}{2}(\cos\beta+\cos 2\beta) & -e^{i(\alpha+2\gamma)}\sin\beta\cos^2\frac{\beta}{2} \\ e^{2i(\alpha-\gamma)}\sin^4\frac{\beta}{2} & e^{i(2\alpha-\gamma)}\sin\beta\sin^2\frac{\beta}{2} & e^{2i\alpha}\frac{\sqrt{6}}{4}\sin^2\beta & e^{i(2\alpha+\gamma)}\sin\beta\cos^2\frac{\beta}{2} & e^{2i(\alpha+\gamma)}\cos^4\frac{\beta}{2} \end{pmatrix}. \tag{A.7}$$

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